Gribov Problem for Gauge Theories: a Pedagogical Introduction

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Abstract

The functional-integral quantization of non-Abelian gauge theories is affected by the Gribov problem at non-perturbative level: the requirement of preserving the supplementary conditions under gauge transformations leads to a non-linear differential equation, and the various solutions of such a non-linear equation represent different gauge configurations known as Gribov copies. Their occurrence (lack of global cross-sections from the point of view of differential geometry) is called Gribov ambiguity, and is here presented within the framework of a global approach to quantum field theory. We first give a simple (standard) example for the $SU(2)$ group and spherically symmetric potentials, then we discuss this phenomenon in general relativity, and recent developments, including lattice calculations.

1

I. INTRODUCTION

In the modern geometrical formulation of field theories one may start with general properties of classical dynamical systems described globally by a trajectory or history [1]. A *history* is a section of a fibre bundle E having the space-time manifold (M, g) as its base space. The typical fibre is known as *configuration space* and can be denoted by C. If the dynamical system consists of a set of fields, such a set takes its values in C. One usually assumes that a fibre bundle E is associated with a group $\mathcal{G}$; such a group can act globally on the fields, or instead it can be a Lie group that acts locally on each fibre. In the latter case, it is more appropriate to consider a principal fibre bundle P [2] which has $\mathcal{G}$ itself as the typical fibre, and a connection on P. The connection on P defines a connection 1-form on the space-time (M, g), known as a *gauge field* and taking values in the Lie algebra $L(\mathcal{G})$ of $\mathcal{G}$. The connection is therefore a collection of sections of trivial bundles $U_i \times T^*(M) \times L(\mathcal{G})$, which are related to each other by group actions in the overlaps $U_i \cap U_j$. These group actions are called *gauge transformations*, and for the whole dynamics to remain invariant under gauge transformations the gauge field itself has to become dynamical. The space Φ of all possible field histories includes both those that do and those that do not satisfy the Euler–Lagrange equations (when there is an action principle for the former), and can be viewed as an infinite-dimensional manifold. To sum up, the space of histories Φ is a principal fibre bundle having the infinite-dimensional Lie group $\mathcal{G}$, the gauge group, as its typical fibre, while physics takes place on the base space of this bundle, the latter being the quotient space $\Phi/\mathcal{G}$ called the space of orbits [1].

On the other hand, the functional-integral quantization of gauge theories of fundamental interactions relies upon the following properties. For a given gauge theory with action functional S and generators of infinitesimal gauge transformations $R^i{}_\alpha$ with associated linearly independent vector fields $\mathcal{R}_\alpha \equiv R^i{}_\alpha \frac{\delta}{\delta \varphi^i}$, one has (hereafter, Greek indices from the beginning of the alphabet are Lie-algebra indices, Greek indices from the middle of the alphabet are space-time indices, while lower case Latin indices are used for fields at a space-time point [1])

$$\mathcal{R}_\alpha S = \mathcal{R}_\beta S = 0 \Longrightarrow \Big[\mathcal{R}_\alpha, \mathcal{R}_\beta\Big] S = \mathcal{R}_\alpha \mathcal{R}_\beta S - \mathcal{R}_\beta \mathcal{R}_\alpha S = 0. \tag{1}$$

If all flows that leave the value of the action invariant can be expressed, at each point of Φ, as linear combinations of the $\mathcal{R}_\alpha$ and skew fields at that point, the most general solution of

Eq. (1) reads as

$$\left[\mathcal{R}_\alpha, \mathcal{R}_\beta\right] = C^\gamma_{\alpha\beta}\, \mathcal{R}_\gamma + \frac{\delta S}{\delta\varphi^j} T^j_{\alpha\beta}. \tag{2}$$

Hereafter, we assume that the tensor fields $T_{\alpha\beta}$ vanish, and that the $C^\gamma_{\alpha\beta}$ are constant, in that they are independent of the fields (but they might depend on space-time coordinates). The $C^\gamma_{\alpha\beta}$ are therefore structure constants of an infinite-dimensional Lie group. Since we are studying a gauge theory, what is crucial is the existence of equivalence classes under the action of gauge transformations. The information on such equivalence classes is encoded into suitable 'coordinates', say (I^A, χ^α), obtained as follows. The I^A are *non-local functionals* which pick out the orbit where the field φ lies, whereas the $\chi^\alpha(\varphi)$ pick out the particular point on the orbit corresponding to the field φ. They are precisely the gauge functionals considered by Faddeev and Popov [3]. The physical $\langle\text{out}|\text{in}\rangle$ amplitude can be expressed, formally, by functional integration over equivalence classes (under gauge transformations) of field configurations:

$$\langle\text{out}|\text{in}\rangle = \int \mu(I)\mathrm{e}^{\mathrm{i}S(I)}\mathrm{d}I. \tag{3}$$

It is now possible, at least formally, to re-express this abstract functional-integral formula in terms of the original field variables, under the assumption (not always verified) that the χ^α coordinates are globally defined. For this purpose, the integration over equivalence classes is made explicit by introducing the χ-integration with the help of a δ distribution, say $\delta(\chi(\varphi) - \zeta)$. After obtaining the Jacobian $J(\varphi)$ of the coordinate transformation from (I^A, χ^α) to φ^I, its functional logarithmic derivative shows that J takes the form $J(\varphi) = N(\varphi)\det\mathcal{F}$, where a careful use of dimensional regularization can be applied to reduce $N(\varphi)$ to a factor depending only on the non-local functionals I^A [1]. Its effect is then absorbed into the measure over the field configurations, so that the $\langle\text{out}|\text{in}\rangle$ amplitude reads as

$$\langle\text{out}|\text{in}\rangle = \int \mathrm{d}\varphi\, \det\mathcal{F}\, \delta(\chi(\varphi) - \zeta)\mathrm{e}^{\mathrm{i}S[\varphi]}. \tag{4}$$

At this stage one performs a Gaussian average over all gauge functionals, and denoting by $\rho_{\alpha\beta}$ a constant invertible matrix, and by σ^α and ψ^β two real-valued and *independent* fermionic fields [1], the functional integral for the $\langle\text{out}|\text{in}\rangle$ amplitude is eventually re-expressed in the form

$$\langle\text{out}|\text{in}\rangle = \int \mathrm{d}\varphi\, \mathrm{d}\sigma\, \mathrm{d}\psi\, \mathrm{e}^{\mathrm{i}[S(\varphi)+\chi^\alpha\rho_{\alpha\beta}\chi^\beta+\sigma_\alpha\mathcal{F}^\alpha_\beta\psi^\beta]}. \tag{5}$$

The physical predictions of the theory are χ- and ρ-independent. The operator $\mathcal{F}^\alpha_\beta$ is called the ghost operator, and the fields σ^α and ψ^β are the corresponding ghost fields [1]. The

3

physical idea is due to Feynman [4], while the development of the corresponding formalism in quantum gravity was obtained by DeWitt [5]. However, to have a well defined functional-integral representation, it is more convenient to start from an Euclidean formulation. In particular, in the one-loop quantum theory, one considers infinitesimal gauge transformations, and if the theory with action S is bosonic, one tries to choose χ^α in such a way that both the ghost operator and the operator on perturbations of the gauge field are of Laplace type [6]. The addition of the gauge-averaging (or gauge-fixing) term $\chi^\alpha \rho_{\alpha\beta} \chi^\beta$ plays a crucial role in ensuring that both the ghost operator and the gauge-field operator have a well defined Green function. Such a term corresponds to the supplementary (or gauge-fixing) condition already occurring in the classical theory [5].

II. ASYMPTOTIC CONDITIONS FOR PURE YANG–MILLS THEORIES

In geometrical language, the Yang–Mills potentials $A_\mu^\alpha(x)$ are components of the pull-back to space-time of the Lie-algebra-valued connection 1-form $A_\mu^\alpha \mathrm{d}x^\mu$ on the principal Yang–Mills bundle, and one has basis matrices G_α for an l-dimensional irreducible representation of the Yang–Mills Lie algebra with structure constants $f_{\beta\gamma}^\alpha$ such that

$$[G_\alpha, G_\beta] = G_\gamma \, f_{\alpha\beta}^\gamma.$$

The gauge potentials A_μ^α and basis matrices G_α can be used to define the matrices $A_\mu \equiv G_\alpha \, A_\mu^\alpha$, and one can write finite gauge transformations in the form

$$^D A_\mu \equiv -D_{,\mu} D^{-1} + D A_\mu D^{-1}. \tag{6}$$

This equation describes the gauge orbit for the configuration A_μ, i.e. the set of all gauge-equivalent configurations corresponding to A_μ. One can further say that finite *little gauge transformations* [1] are the proper subset of transformations described by Eq. (6) such that the matrix D can be written as

$$D = \mathrm{e}^{G_\alpha \xi^\alpha}, \tag{7}$$

where the ξ^α are finite functions on space-time that vanish at spatial infinity. This condition implies that D tends to the identity therein. The matrices D are the representation matrices for the group $\mathcal{G}$ generated by the G_α.

Big gauge transformations are defined by Eqs. (6) and (7), supplemented by the following asymptotic conditions on the D matrices (the ξ^α functions no longer vanish at spatial infinity):

$$D(x) \to \pm 1_l \text{ as } x \to \infty, \tag{8}$$

1_l being the l-dimensional unit matrix. In this case, both the identity and minus the identity are necessary, because the transformation law (6) does not change on replacing D by $-D$.

More precisely, the infinite-dimensional Lie group $\mathcal{G}$ with structure constants $C^\gamma_{\alpha\beta}$ as in Eq. (2), with representation matrices expressed by Eq. (7), is the *proper gauge group* (its elements are the little gauge transformations defined above). The *full gauge group* is obtained by adjoining to the proper gauge group all transformations of the space Φ of histories into itself, independent of field variables, which leave the action functional S un-affected and do not result from global symmetries. When big gauge transformations exist, both the full gauge group and the space of histories have disconnected components.

Note that both big and little gauge transformations do not affect the dynamical variables at infinity. This leads to an investigation of the boundary conditions that should be imposed upon the dynamical variables. In general, the Yang–Mills fibre bundle might be non-trivial in spacelike directions. The possible physical implications of such a non-trivial nature have not yet been studied, and we shall assume hereafter that the space-time manifold is asymptotically Minkowskian, and that non-trivial aspects can only occur in the time direction [1].

On focusing on a four-dimensional space-time, one requires that the components of the Yang–Mills potential should have the following asymptotic behaviour [1]:

$$A^\alpha_0 \sim \frac{1}{r^{1/2+\varepsilon}}, \qquad A^\alpha_i \sim \frac{1}{r^{1+\varepsilon}}, \tag{9}$$

where ε is an arbitrarily small positive number. Moreover, the time derivatives of A^α_i are taken to satisfy the fall-off condition

$$A^\alpha_{i,0} \sim \frac{1}{r^{3/2+\varepsilon}}, \tag{10}$$

which implies that the field strength defined as [1]

$$F^\alpha_{\mu\nu} \equiv A^\alpha_{\nu,\mu} - A^\alpha_{\mu,\nu} + f^\alpha_{\beta\gamma} \, A^\beta_\mu \, A^\gamma_\nu \tag{11}$$

5

satisfies the fall-off conditions

$$F_{i0}^{\alpha} \sim \frac{1}{r^{3/2+\varepsilon}}, \qquad F_{ij}^{\alpha} \sim \frac{1}{r^{2+\varepsilon}}. \tag{12}$$

Three properties are therefore found to hold [1]:

(i) All physically admissible solutions of the field equations are included in the space Φ of field histories, in particular those having non-vanishing total charge. For these solutions, the charge integral converges.

(ii) The spatial integral of the energy density T^{00} is finite, whether or not the field strength $F_{\mu\nu}^{\alpha}$ in the expression of the energy-momentum tensor $T^{\mu\nu}$ satisfies the field equations.

(iii) The spatial integral of the Yang–Mills Lagrangian is finite.

The above conditions are compatible with Eq. (8), but add a more accurate requirement: one should require the conditions

$$D \mp 1_l \sim \frac{1}{r^{\varepsilon}}, \qquad D_{,0} \sim \frac{1}{r^{1/2+\varepsilon}}. \tag{13}$$

The space Φ consists therefore of all bundle structures of field histories $A_{\mu}^{\alpha}(x)$ satisfying the asymptotic conditions (9) and (10) in every patch of the fibre bundle. The gauge group $\mathcal{G}$, which is the typical fibre of Φ, can be identified with the set of all matrix functions, having the exponential form in Eq. (7) and satisfying the asymptotic conditions (8) and (13). The transition functions between patches of the bundle are matrix functions of the same form, restricted to overlapping regions. Last, note that the vanishing trace of the generators G_{α} implies that $\det D(x) = 1$ for all x.

In the Hamiltonian formalism, a frequently used supplementary condition is the *axial gauge*. The resulting ghost operator reads as [1]

$$\mathcal{F}_{\beta'}^{\alpha} = -\delta_{\beta}^{\alpha}\delta_{,1}(x,x') - f_{\gamma\beta}^{\alpha}A_1^{\gamma}\delta(x,x'), \tag{14}$$

the Green function of which, with indices suppressed for simplicity, is

$$\begin{aligned} G(x,y) &= \left[a\theta(x^1,y^1) + (a-1)\theta(y^1,x^1)\right]\delta(x^0,y^0)\delta(x^2,y^2)\delta(x^3,y^3) \\ &\quad \times \mathcal{P}\exp\left[-\int_{y^1}^{x^1} f_{\alpha}A_1^{\alpha}(x^0,z^1,x^2,x^3)\mathrm{d}z^1\right], \end{aligned} \tag{15}$$

where a is any real number, and $\mathcal{P}$ denotes path ordering [1]. Moreover, the f_α are matrices built out of the structure constants $f^\beta_{\alpha\gamma}$, such that

$$[f_\alpha, f_\beta] = f_\gamma \, f^\gamma_{\alpha\beta},$$

and they generate the adjoint representation of the gauge group [1].

One of the apparently attractive features of these Green functions is their local nature in the time variable, i.e. ghost fields do not propagate. However, the axial gauge is incompatible with the asymptotic conditions (9), (10), (12), (13) which are necessary to define the space of field histories and the gauge group with a certain control of the limiting behaviour at spatial infinity.

To prove the latter statement, let us consider the standard gauge-fixing condition $\chi^\alpha = 0$. One can assume that, to fulfill such a condition, one has only to perform a finite gauge transformation (see Eq. (6)) such that

$$0 = {}^D A_1 = -D_{,1} \, D^{-1} + D \, A_1 \, D^{-1},$$

or, equivalently, $D_{,1} = D \, A_1$. This equation can be solved on each patch of the fibre bundle, by simply integrating with the condition that $D = 1_l$ at $x^1 = -\infty$. In general, however, D does not wind up equalling the identity at $x^1 = \infty$ for all x^0, x^2 and x^3, even if the potentials A^α_μ satisfy the asymptotic conditions (9). The same occurs when the χ^α are set equal to any other functions of the x^μ variables that vanish at spatial infinity [1].

For a supplementary (or gauge) condition to be compatible with the asymptotic conditions (9), (10), (12), (13) the resulting Green functions of ghost fields should be non-local in the time variable. For example, the temporal gauge $A^\alpha_0 = 0$ is instead compatible with the asymptotic conditions given above [1].

III.   GRIBOV PHENOMENON

The careful assignment of asymptotic conditions on gauge potentials is justified by the analysis of global properties of the latter. In the case of Yang–Mills theories, a new phenomenon is found to occur with respect to quantum electrodynamics.

Since the discovery of instantons [7], the interest in global properties led to the attempt of understanding quark confinement as well by virtue of such global properties [8,9], while

perturbative methods failed to provide any clue on such aspects of gauge theories. Among the peculiar features of global nature, the *Gribov ambiguity* deserves special attention. In the simplest possible terms, on requiring that the supplementary (or gauge-fixing) condition should be preserved under gauge transformations of the potential, one obtains a non-linear differential equation, *for each choice of gauge-fixing compatible with the asymptotic conditions* (9), (10) and (13). The key remark is now that, on the space of orbits $\mathcal{O} \equiv \Phi/\mathcal{G}$ which is the base space of Φ, the choice of gauge-fixing on the potentials:

$$\chi[A] - \zeta = 0$$

describes a certain surface which intersects [10] the orbits of Eq. (6). In the very definition of gauge fixing at perturbative level, one requires that each orbit should intersect the gauge-fixing surface only once, so as to ensure uniqueness of the potentials satisfying dynamical equations for a given choice of gauge fixing. In the attempt of extending such a procedure globally all over $\mathcal{O}$, however, one faces difficulties resulting from the non-trivial topology of the fibre bundle in the non-Abelian case [11]. Thus, the gauge-fixing surface intersects some orbits more than once. In other words, in the overlapping regions among various patches of the bundle, transition functions are not single-valued; they might remain single-valued, however, within each topological sector. *If* this is the case, the perturbative evaluation of in-out amplitudes through the gauge-fixed functional integral is justified. By contrast, at global level, transition functions may take different values in the various topological sectors, by virtue of particular gauge transformations which relate them. By virtue of these *big gauge transformations*, two potentials close to each other in a given patch with respect to the metric on the space of histories [1]

$$\gamma_{\alpha\beta'}^{\mu\nu'} \equiv K_{YM}^{-2} \sqrt{-g} \; \gamma_{\alpha\beta} \; g^{\mu\nu} \delta(x,x'),$$

where K_{YM} is the Yang–Mills coupling constant, $\gamma_{\alpha\beta} \equiv -\text{tr}(f_\alpha f_\beta)$ is the Cartan–Killing metric and $g^{\mu\nu}$ is the contravariant form of the space-time metric tensor, can turn out to be quite far apart in another patch, since one of the two can now lie in a different topological sector.

Indeed, the Gribov phenomenon can be viewed as a topological obstruction to achieving continuity in the choice of values ζ^α taken by the gauge-fixing functional χ^α. In geometrical language, one can say that it is impossible to find a global cross-section for the Yang–Mills

8

fibre bundle. Singer [12] has indeed proved that, for $SU(N)$ groups with base manifold S^3 or S^4, no gauge-fixing condition exists which is compatible with the asymptotic conditions (9), (10), (12), (13). Since the more involved symmetry groups admit $SU(2)$ as a sub-group, the above statement holds for all physically relevant gauge theories. It should be pointed out that, in the case of quantum chromodynamics, two particular gauge-fixing conditions exist for which the Gribov ambiguity does not seem to occur: the axial and temporal gauges. However, the former does not satisfy the asymptotic conditions (9), (10), (12), (13) as we have seen at the end of Sec. II, while in the latter the Gauss law does not automatically hold. If the Gauss law is also enforced, one again finds the occurrence of Gribov copies [13].

Along the lines of the original Gribov argument [14], let us consider a non-Abelian theory with potentials subject to the infinitesimal gauge transformations

$$\delta A^\alpha_\mu \equiv \delta^\alpha_\beta \partial_\mu \lambda^\beta + f^\alpha_{\gamma\beta} \, A^\gamma_\mu \, \lambda^\beta \equiv D^\alpha_{\mu\beta} \, \lambda^\beta. \tag{16}$$

Moreover, let us impose a covariant gauge condition such as the Lorenz gauge for Yang–Mills theory:

$$\partial^\mu A^\alpha_\mu = 0, \tag{17}$$

for potentials satisfying the asymptotic conditions (9) and (10). The general field variables φ^i of Sec. I are now the Yang–Mills potentials $A^\alpha_\mu(x)$, and Eq. (17) is an example of gauge choice implemented, in the path integral (4), through the Dirac distribution $\delta(\chi(\varphi) - \zeta)$. The requirement of preserving the gauge-fixing condition (17) under gauge transformations yields

$$0 = \partial^\mu \left(A^\alpha_\mu + \delta A^\alpha_\mu \right) - \partial^\mu A^\alpha_\mu = \partial^\mu D^\alpha_{\mu\beta} \, \lambda^\beta = \mathcal{F}^\alpha_\beta \, \lambda^\beta. \tag{18}$$

This can be seen as an equation defining the singularities of the ghost operator, i.e. its zero-modes. More precisely, one can interpret Eq. (18) as an eigenvalue equation for the ghost operator with vanishing eigenvalue. If the corresponding eigenfunction is non-trivial, it is said to be a *zero-mode*, and the ghost operator becomes singular. A physics-oriented interpretation is instead as follows: there exist a number of gauge potentials, related to each other by a gauge transformation, and satisfying the dynamical equations with a given choice of supplementary (or gauge-fixing) condition, when the ghost fields ψ^β have zero-mass bound states. The latter interpretation is suggested by the fact that $\mathcal{F}^\alpha_\beta$ occurs in the action functional of the theory only through the term $i\sigma_\alpha \mathcal{F}^\alpha_\beta \psi^\beta$ in Eq. (5). Equation (18)

is therefore equivalent to having

$$\mathcal{F}^{\alpha}_{\beta}[A_\mu]\psi^\beta = \varepsilon\,\psi^\alpha, \text{ with } \varepsilon = 0, \tag{19}$$

where the parameter λ^β has been replaced by the ghost field ψ^β.

On assuming, for simplicity, that the gauge potential vanishes before performing the gauge transformation, Eq. (19) becomes, for a generic value ε of the ghost eigenvalue,

$$\partial^\mu \partial_\mu \, \delta^{\alpha}_{\beta} \, \psi^\beta = \delta^{\alpha}_{\beta} \, \square\, \psi^\beta = \varepsilon\,\psi^\alpha. \tag{20}$$

This equation can only be solved for non-negative values of ε. On taking increasingly large magnitudes of the potentials, i.e. on moving away from the neighbourhood of $A_\mu = 0$ where perturbative calculations are performed for Yang–Mills theories, one arrives at a sufficiently large magnitude of A_μ for which there exist solutions of Eq. (19) with $\varepsilon = 0$, i.e. Eq. (18) is satisfied. On further increasing the magnitude of the potentials one finds negative eigenvalues, until for an even bigger value of such a magnitude one finds again a solution of Eq. (19) with $\varepsilon = 0$.

Note that Eq. (20) is, at this stage, an hyperbolic equation having non-vanishing solutions also in the limiting Abelian case; a way to get rid of non-uniqueness of the solutions of the equation expressing preservation of the gauge-fixing is to perform a Wick rotation and hence consider an Euclidean space. In such a space one can understand the behaviour of the solutions of Eq. (20) by exploiting index theory and the concept of spectral flow (see Ref. [15] and references therein).

One can thus imagine that the potentials solving Eq. (18) divide the space of potentials into regions, in each of which Eq. (20) has a certain number of eigenvalues, corresponding to bound states of the ghost field ψ^β. In correspondence to the surfaces defined by the potentials satisfying Eq. (18), known as *Gribov horizons*, which separate the various *Gribov regions*, there exist massless ghost states. The first Gribov region C_0 has no ghost bound states, the second Gribov region C_1 has one ghost bound state, the k-th Gribov region C_k has $(k-1)$ ghost bound states.

Since the potentials occurring in the Gribov regions C_n, $n = 1, 2, 3\ldots$ are Gribov copies of the configurations occurring in C_0, as Gribov himself did show in his original paper, the functional integration should be restricted to C_0. Two kinds of Gribov copies can be found, i.e. those obtained from the potentials in C_0 from *big gauge transformations*, resulting

10

from topological effects previously described, as well as equivalent copies of the potentials in C_0 that exist within C_0 itself. While the former can be simply ignored in perturbative calculations, the latter make it necessary to use a greater care. To avoid overcounting copies in the functional integral, it is necessary to further restrict integration to a domain known as the *fundamental modular region* [16], the boundary of which is studied in Refs. [16-18], and within which the Gribov ambiguity no longer occurs [19].

In the original analysis by Gribov [14], the restriction of the integration region in configuration space has the effect of removing the infrared singularity of perturbative theory and leads to a linear increase of the interaction among non-Abelian charges at large distances, i.e. a possible mechanism of quark confinement.

IV. GRIBOV EQUATION

Before further discussing the implications of the Gribov analysis, it is appropriate to present a simple example, used by Gribov himself, of the occurrence of such a phenomenon. An explicit solution of the equation expressing preservation of the Coulomb gauge-fixing condition at $A_\mu = 0$ under gauge transformations:

$$\partial_r((\partial^r D)D^{-1}) = 0, \tag{21}$$

can be found for the $SU(2)$ group, which can be easily parametrized because the exponential map on its Lie algebra is in one-to-one correspondence with the interior of the smallest circle where all points map to $-I$. Thus, every element with the exception of minus the identity has a unique representation U:

$$U = e^{i\vec{n}\cdot\vec{\sigma}} = I\cos|\vec{n}| + i\hat{n}\cdot\vec{\sigma}\sin|\vec{n}|, \tag{22}$$

where $\hat{n} \equiv \vec{n}/|\vec{n}|$ and $\vec{n}\cdot\vec{n} < \pi$. To obtain an element $U(\vec{x})$ of the gauge group, one has simply to make $\vec{n}$ into a function of $\vec{x}$. Let us now assume, to obtain radial symmetry and the correct asymptotic behaviour, that [20]

$$\vec{n} = \frac{\omega(r)}{r}(y, x, z), \tag{23}$$

where the exchange of x and y is intended, and let us pass to spherical polar coordinates, obtaining

$$U(r, \theta, \phi) = \begin{pmatrix} c_\omega + ic_\theta s_\omega & e^{i\phi}s_\theta s_\omega \\ -e^{-i\phi}s_\theta s_\omega & c_\omega - ic_\theta s_\omega \end{pmatrix} \tag{24}$$

11

where s_α and c_α denote $\sin\alpha$ and $\cos\alpha$, respectively. To obtain Eq. (24) one has to consider the Jacobian matrix for the coordinate transformation and the standard row $\times$ column rule for the scalar product $\hat{n}\cdot\vec{\sigma}$. On using the familar expressions of the divergence and gradient operators in polar coordinates, and the further change of coordinate $t \equiv \log(r)$, Eq. (21) on U yields [20]

$$\left(U_{tt} + U_t + U_{\theta\theta} + \frac{c_\theta}{s_\theta}U_\theta + \frac{1}{s_\theta^2}U_{\phi\phi}\right)U^\dagger + U_t U_t^\dagger + U_\theta U_\theta^\dagger + \frac{1}{s_\theta^2}U_\phi U_\phi^\dagger = 0, \qquad (25)$$

where subscripts denote derivative with respect to the variable expressed. In the light of Eq. (24), Eq. (25) takes the form

$$\begin{pmatrix} ic_\theta & e^{i\phi}s_\theta \\ -e^{-i\phi}s_\theta & -ic_\theta \end{pmatrix}\left(\ddot{\omega} + \dot{\omega} - 2\sin\omega\cos\omega\right) = 0, \qquad (26)$$

where a dot denotes derivative with respect to t. On passing to the variable $\overline{\omega} \equiv 2\omega$, and leaving aside the 2×2 matrix in Eq. (26) (since it is independent of $\overline{\omega}$), one finds eventually a differential equation describing a damped pendulum in a constant gravitational field, with $\overline{\omega}$ being the angle from the point of unstable equilibrium. Since $U = I$ at the origin, one finds $\overline{\omega} = 2k\pi$, where k is an integer. Moreover, since $\overline{\omega}$ is determined up to a multiple of 2π, there is no loss of generality in taking $k = 0$. One therefore obtains what is called the *Gribov pendulum equation*

$$\left(\frac{d^2}{dt^2} + \frac{d}{dt} - 2\sin\right)\overline{\omega} = 0, \qquad (27)$$

with the following boundary (or asymptotic) condition:

$$\lim_{t\to-\infty}\overline{\omega}(t) = 0. \qquad (28)$$

There exist three solutions with the initial condition (28): either the pendulum never falls, or it falls anti-clockwise, or instead clockwise.

An analogous treatment of the Gribov pendulum equation, which however does not rely upon the matrix representation, can be found in Ref. [21]. At a deeper level, *more than a solution is found because the whole of configuration space is being explored*. On restricting the analysis to a bounded region, as it always occurs in perturbation theory, Gribov copies are instead ignored.

V. GRIBOV AMBIGUITY AND GENERAL RELATIVITY

In Sec. II, on discussing the asymptotic conditions for gauge theories, we focused our attention on Yang–Mills theories. Actually, the manifestly covariant formalism can treat gravitation as well in the same language used for non-Abelian gauge theories. It may be interesting to investigate a possible occurrence of the analogue of Gribov ambiguity in general relativity. Several ways to express Einstein's theory as a gauge theory have been proposed in the literature; the matter is that one can treat this theory just in the connection formalism adopted in the introduction for any dynamical system.

General relativity can be mathematically formulated starting from a pseudo-Riemannian 4-manifold $\mathcal{M}$ endowed with an atlas of coordinate charts. Symmetries usually considered are transformations of the metric tensor $g_{\mu\nu}$ induced by general coordinate transformations, hence the passive point of view of the diffeomorphism group is preferred to the active one. The biggest group Q of passive dynamical symmetries of Einstein's equations is the group of transformations reading as [22]

$$x'^{\mu} = f^{\mu}(x^{\nu}, g_{\rho\sigma}(x)).$$

It is larger than the general coordinate transformation group, usually considered, which is a non-normal subgroup of Q [23].

In a gauge theory of gravitation one can adopt a principal fibre bundle which has, as base space, the union of equivalence classes of all metric tensors, solutions of Einstein's equations, generated by passive diffeomorphisms [23]. Despite previous considerations about a more general symmetry group involved, such a choice of principal bundle is sufficient to characterize the theory. The quotient group where dynamics is reduced, in fact, is the same for the three symmetry groups: the passive diffeomorphisms, the active ones or Q. The elements of this quotient group are the gauge orbits of general relativity. It is natural to investigate topological properties in this case, seeking for the existence of global cross-sections, whose lack is found in Yang–Mills theory.

The differences from the latter theory result from the nature of the diffeomorphism group, which is a non-analytic infinite-dimensional Lie group, while Yang–Mills symmetry groups are Baker–Campbell–Hausdorff groups [24]. The Gribov ambiguity arises from the non-trivial topology of gauge orbits in Yang–Mills theory; in general relativity their topology is even less regular, so that the problem becomes more involved as well as more fundamental.

Mathematical complications arise from the nature of the diffeomorphism group. Yang–Mills theory is a field theory on a background space-time, and the action of the gauge group in that case is on a inner space. In general relativity, instead, the action of the gauge group is an extension to tensors over space-time of the diffeomorphisms of space-time itself [25].

It is very difficult, in the manifestly covariant configuration space approach, to face the Gribov problem in general relativity. Indeed, one would need tools to make a clean separation between gauge variables and a basis of gauge-invariant observables. In other words, it is not simple to find suitable coordinates (I^A, χ^α) as in Sec. I. The Hamiltonian formulation has, at least locally, a natural tool for such a separation, i.e. the Shanmugadhasan canonical transformations [26]. In this approach, the singularity of Lagrangians, both in particle physics and in general relativity, makes it necessary to use the Dirac–Bergmann theory [27] of constrained Hamiltonian systems. Since it is the analogue of the space of gauge orbits in the global approach, only the constraint submanifold of phase space has physical relevance.

In phase space there are as many arbitrary Hamiltonian gauge variables as first-class constraints, which are the generators of the Hamiltonian gauge transformations. They may be used as the I^A coordinates of Sec. I, hence they determine a coordinatization of the gauge orbits inside the constraint submanifold. To obtain the reduced phase space, one has to add as many gauge-fixing constraints as first-class ones. As a consequence of imposing gauge-fixing constraints, first-class constraints are turned into the second class and therefore treated with the help of Dirac brackets (such constraints are then strongly vanishing).

The Dirac observables, which in general can be non-local, only give a coordinatization of the physically relevant space. On the other hand, it is the analysis of its topological properties that makes it possible to say whether a given dynamical system (with constraints) admits a subfamily of globally defined Shanmugadhasan canonical transformations. The existence of such a subfamily is the same as the one of a global cross-section, from the point of view of differential geometry: it means that the system admits preferred global separations between gauge and observable degrees of freedom. In other words, in the Hamiltonian approach the existence of globally defined Shanmugadhasan canonical transformations makes it possible to avoid the Gribov ambiguity.

Actually, the existence of these transformations has not yet been proved. Indeed, important constraints like the Yang–Mills Gauss law and the ADM supermomentum constraints [25] are partial differential equations of elliptic type, hence they may admit zero modes

according to the choice of function space. One has to solve the same kind of problems previously treated about the ghost operator in the global approach.

To obtain, at least as a first approximation, the main non-topological properties of a system, the Hamiltonian approach needs to avoid the analogue of Gribov ambiguity in general relativity. In fact, in this formalism one assumes that all fields have to belong to suitably weighted Sobolev spaces so that the allowed spacelike hypersurfaces are Riemannian 3-manifolds without Killing vectors.

We may conclude that in general relativity one has to face the analogue of the Gribov problem; in this theory it is more difficult to arrive at some conclusion by virtue of highly non-trivial topology of the diffeomorphism group.

VI. RECENT DEVELOPMENTS IN CONTINUUM GAUGE THEORIES

We have seen in Sec. II that, to control the asymptotic behaviour of gauge potentials, one has to impose appropriate boundary conditions. The choice of these conditions rules the occurrence of Gribov copies. However, one could make a distinction between two different cases of ambiguity of gauge potentials, according to which kind of boundary conditions are imposed. In this way one can have a weak Gribov problem and a strong one [28].

When there exist always some configurations which have copies, we can talk of the *weak Gribov problem*. This phenomenon, which is found in any regular gauge, occurs even on a compact space and inside a given topological sector of configuration space.

When there are copies of the vanishing configuration ($A_\mu = 0$), hence pure gauge potentials satisfying the gauge condition, we can talk of the *strong Gribov problem*. On choosing weak-decay boundary conditions, we might find the occurrence of this phenomenon in an infinite space. On the other hand, it is not customary inside a given topological sector in a compact space; the same occurs on choosing strong decay conditions in a non-compact space.

In terms of functional integration the weak problem is less harmful than the strong one. In fact, one might view the strong Gribov ambiguity as a lack of strict positivity for the functional measure, and the weak one as a lack of monotonicity. For smooth configurations on a smooth compact four-dimensional space, the strong Gribov problem is not expected.

We have seen that boundary conditions are fundamental in dealing with the Gribov am-

biguity. Indeed, appropriate boundary conditions (or compactification) may be imposed to avoid the strong Gribov problem, as is usually done in the constructive study of the ultraviolet limit of non-Abelian gauge theories to avoid the infrared problem. The weak Gribov phenomenon, instead, cannot be avoided [28]. Anyway, in perturbative gauge theories, a topological analysis shows that the Gribov problem is irrelevant [29].

One could manage to get rid of the Gribov ambiguity by employing different theories, that generalize the usual gauge theories. For generalized connections [30], the Gribov problem is completely irrelevant for the calculation of functional integrals, hence for the $\langle$out$|$in$\rangle$ amplitude, on assuming absolute continuity of the considered measure on the space of Ashtekar's generalized connections [31] with respect to the induced Haar measure.

Up to this limitation, the Gribov phenomenon is still present with generalized connections, but it is not a problem. Indeed, we have found that topological non-triviality of the configuration space is responsible for the occurrence of Gribov ambiguity; this non-triviality is concentrated on a zero-measure subset [30] in this formalism.

Among several techniques proposed to take care of Gribov copies, one might use a quantization which does not require gauge fixing [32] such as a generalized Gupta–Bleuler method [33], one can modify either the gauge-fixing procedure [34] or the functional integration [35], otherwise using non-local variables such that gauge fixing is no longer required [36].

Another recently proposed way to get rid of the Gribov ambiguity is a restriction on the norm of the $\langle$out$|$in$\rangle$ amplitude, not considering its phase. In this way Yang–Mills theory would be free from ambiguities if used for the description of observables [37].

The most promising way to avoid the Gribov ambiguity in QCD is, as far as we can see, the stochastic quantization approach [38], in which, although there are Gribov copies, they have no influence on expectation values. This approach determines an Euclidean probability distribution directly in configuration space, i.e. the space of gauge potentials, without reduction to the orbit space.

The first Gribov region may be characterized as the set of relative minima with respect to local gauge transformations of the minimizing functional $F_A[D] \equiv ||^D A||^2$, analogue of the ghost operator $\mathcal{F}$ of Sec. III. The fundamental modular region Λ, instead, may be characterized as the set of absolute minima. The latter is free of Gribov copies, apart from the identification of gauge-equivalent points on the boundary $\partial\Lambda$, and may be identified with the gauge orbit space.

One can clearly say that in continuum gauge theories the Gribov ambiguity is still an open problem at non-perturbative level. In this respect, one should here mention an important result of Neuberger [39], according to which the action used in BRST quantization may engender vanishing partition functions, by virtue of Gribov copies which cancel each other's effects, and the same may occur for expectation values of physical operators.

Apart from future developments of the previous methods or other ones to get rid of the Gribov phenomenon, for the time being some conclusions may be outlined about implications of the existence of Gribov copies in continuum gauge theories.

First, in the high-energy sector of non-Abelian gauge theory, the restricted domain of integration in the functional amplitude has surprising consequences: they contradict the expected standard behaviour of the theory [28]. Moreover, the Gribov problem leads to a cutoff on frequencies stronger than the expected one, resulting from standard perturbative asymptotic freedom [28].

In the infrared sector, Gribov himself in his original paper argued that this ambiguity leads to an effective infrared cutoff; in fact, he related this behaviour with confinement. Last, we may conclude by saying that in non-Abelian theory the Gribov phenomenon seems to be an obstruction, at non-perturbative level, to the coexistence of too many energy scales [28].

VII.   THE CASE OF LATTICE GAUGE THEORIES

In the formulation of lattice gauge theories [40], the gauge-fixing procedure and the generation of Gribov copies, obtained via Monte Carlo simulations and, in particular, via random gauge rotations [41], are by now well-established techniques, although they have been initially ignored since it is not strictly necessary to fix a supplementary (or gauge) condition in such theories. The gauge-fixing procedure, however, is particularly convenient in some circumstances, and these have led to an investigation of the Gribov problem in this framework as well [42].

The regularization provided by a lattice [43], in fact, makes the gauge group compact, so that the Gibbs average of any gauge-invariant quantity is well-defined. However, asymptotic freedom causes the continuum limit to be the weak coupling limit, and such an expansion requires gauge fixing. Moreover, to extract non-perturbative results from Monte Carlo sim-

ulations, one can use the evaluation of quark/gluon matrix elements, which again requires gauge fixing. The same procedure is also used in smearing techniques.

A brief discussion of the lattice theory formalism may be useful to understand why Gribov copies also occur in lattice gauge theory. One can expect their presence by virtue of the Killingback analysis [44], which is analogous to the investigation of Singer, but is made for Euclidean gauge theories with periodic boundary conditions.

An action describing fermions and gauge fields on the lattice is obtained by replacing differentials with finite differences and constructing a suitable covariant derivative. The theory without fermions is known as "pure gauge theory". The pure gauge action on the lattice is usually taken to be [45]

$$S = \beta \sum_p \left(1 - \frac{1}{N_C} \operatorname{Re} \operatorname{Tr} U_p \right), \tag{29}$$

where $N_C = 2$ for $SU(2)$, and the *plaquette variable* U_p is the ordered product of four link variables in a square. The link variables U_{ij} are the degrees of freedom in the theory, and are elements of the gauge group. A link between two nearest-neighbour sites i and j of the lattice is associated with each link variable U_{ij}. Such a variable is also often written as $U_\mu(x)$, where x denotes the site of the link and μ the direction.

Elements of the gauge group are the gauge transformation variables too, which live on the lattice sites themselves. A gauge transformation acts on a link variable as follows:

$$U'_{ij} = D_i \, U_{ij} \, D_j^{-1}. \tag{30}$$

On using the fundamental representation for gauge fields, the action (29) is known as the *Wilson action*. The first term in the expression is a constant and is added in order to reduce the action to the Yang–Mills one in the classical continuum limit. This limit is taken by letting the lattice spacing go to zero after identifying

$$U_{ij} = \exp\left[-(x^j - x^i)_\mu \, A^\mu \left(\frac{x^j + x^i}{2} \right) \right]. \tag{31}$$

Gauge fixing on the lattice requires to minimize the functional

$$F_L[D] = -\sum_{\mu,x} \operatorname{Re} \operatorname{Tr} D(x) \, U_\mu(x) \, D^\dagger(x + \hat{\mu}), \tag{32}$$

as a function of all gauge transformations $D(x)$. The sum over μ runs from 1 to 3 for the Coulomb gauge, and from 1 to 4 for the Landau gauge. Note that this function is the lattice analogue of the functional $F_A[D] = ||^D A||^2$.

On expanding $U_\mu(x)$ according to the formal identity [46]

$$U_\mu(x) = \exp(-a\,A_\mu(x)) = 1 - a\,A_\mu(x) + a^2 A_\mu^2(x) + \dots, \qquad (33)$$

where a is the lattice spacing, one gets

$$F_L[1] = -\sum_{\mu,x} \mathrm{Tr}\,(1 + a^2 A_\mu^2(x)) = \mathrm{const.} + a^2 F_A[1], \qquad (34)$$

where use has been made of the identity $\mathrm{Tr}\,A_\mu = 0$. Hence, it is clear that minimizing $F_L[D]$ is the same as minimizing $F_A[D]$, in the classical continuum limit.

The functional $F_L[D]$ can be seen as the Hamiltonian of a spin glass system [47]. Such a system can behave in a highly chaotic way, hence giving rise to a large number of local minima. Indeed, such local minima, which correspond to Gribov copies in the continuum gauge theory, appear on the lattice, both for Abelian and non-Abelian fields, in the Coulomb gauge as well as in the Landau gauge. The first numerical evidence for lattice Gribov copies was presented in the early nineties [41], [47], and [48]. The number of copies in the Coulomb gauge was found to be larger than in the Landau gauge [48].

The effects on the evaluation of gauge-dependent quantities resulting from the presence of Gribov copies are studied by comparing the results for quantities, such as gluon and ghost propagators [49], using two different averages: the average considering only the absolute minima, which should give the result in the minimal Landau gauge; and the average considering only the first gauge-fixed copy generated for each configuration. If Gribov copies were not considered, one would obtain the latter average as the result.

A very interesting analysis is provided by the evaluation of the axial current renormalization constant Z_A [41]. The relevance is here twofold: on the one hand, Z_A can be obtained from chiral Ward identities in two distinct ways, either gauge independent (it consists in evaluating matrix elements between hadron states) or gauge-dependent (which consists in evaluating matrix elements between quark states). This engenders an explicit gauge-invariant estimate of Z_A, without any Gribov noise (in that the Gribov copies consist of fluctuations giving rise to a background noise), which can be directly contrasted with the gauge-dependent estimate affected by Gribov copies. On the other hand, there is also the advantage of finding Z_A by solving an algebraic equation of first degree for each time section of the lattice, hence avoiding the systematic errors usually emerging from an exponential fit of decay signals.

The corresponding analysis shows that Gribov copies have visible effects, that can be detected, for example, on looking at the slightly different estimates of Z_A as a function of a parameter. The Gribov fluctuations, however, i.e. those induced from the choice of a particular Gribov copy, are small and do not prevail on the statistical uncertainty. In other words, numerical effects of Gribov copies on a lattice can be divided into two categories: *measurement distortion* and *lattice Gribov noise*.

In most investigations of the influence of Gribov copies on lattice quantities, it has been found that Gribov noise is of the same order as the numerical accuracy of the simulations, and that it scales down as a pure statistical error. In some particular cases Gribov noise seems to be quite large. In maximally Abelian gauge it introduces a clear bias on the number of monopoles [50] and on the value of the Abelian string tension [51,52].

A typical example of distorsion resulting from Gribov copies is the measurement of photon propagator in compact $U(1)$ lattice gauge theories. It has been shown numerically [53–56], that the photon propagator in the Coulomb phase is strongly affected by Gribov noise and that only averages taken on absolute minima of the minimizing functional reproduce the theoretical predictions. Thus, one expects that the same might happen for the gluon and ghost propagators. Analogue results were found in pure $SU(2)$ lattice gauge theory in the minimal Landau gauge [43].

In some circumstances, in lattice too there exist copies which do not seem to be related by a change in the winding index or from a singular local gauge transformation; they are called non-topological Gribov copies [54].

Gribov copies seem to increase in number with increasing lattice volume; experimental data, instead, suggest that some Gribov copies are lost as β is increased, at fixed volume. This is the behaviour that one would expect: a large lattice volume, in fact, is related with a large number of gauge transformation variables $D(x)$. Of course, when $F_L[D]$ depends on more variables, it will have more local minima. Moreover, small β means large coupling [46]; in the continuum limit we may expect Gribov copies in such a situation. To obtain the continuum limit of the theory we have to let both parameters go to infinity, hence it appears highly plausible that the phenomenon of Gribov copies described by lattice theories should occur in the continuum limit as well [47].

It seems fair enough to say, however, that even for lattice theories no undisputable results exist as yet. Investigation of the Gribov problem on a lattice has shown that ambiguities

resulting from the gauge-fixing procedure exist for both Abelian and non-Abelian theories [57] in the Coulomb and Landau gauges. For example, although on the one hand there is evidence that Gribov copies exist in the $SU(3)$ Landau gauge [47], on the other hand the measurement of the $SU(3)$ gluon propagator obtained with the help of numerical simulations performed in the Landau gauge does not point out the existence of the Gribov ambiguity [43].

Anyway, every lattice calculation has to face the Gribov problem, for example by employing stochastic gauge fixing, as is the case for lattice evaluation of gluon screening masses [58]. Furthermore, on looking at the foundations of non-perturbative QCD, if one takes locality and BRST symmetry as guiding principles [59], one again finds the unavoidability of Gribov copies, which are instead absent at the price of dealing with non-local field theory [10].

Acknowledgments

[1] B.S. DeWitt, *The Global Approach to Quantum Field Theory* (Oxford University Press, Oxford, 2003).

[2] T.T. Wu and C.N. Yang, Phys. Rev. **D12**, 3845 (1975); M.F. Atiyah, *Geometry of Yang–Mills Fields*, Lezioni Fermiane, Accademia Nazionale dei Lincei & Scuola Normale Superiore, Pisa (1979).

[3] L.D. Faddeev and V.N. Popov, Phys. Lett. **25B**, 29 (1967).

[4] R.P. Feynman, Acta Phys. Pol. **24**, 697 (1963).

[5] B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon & Brach, New York, 1965); B.S. DeWitt, Phys. Rev. **162**, 1195 (1967).

[6] I.G. Avramidi and G. Esposito, Commun. Math. Phys. **200**, 495 (1999).

[7] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Y.S. Tyupkin, Phys. Lett. **59B**, 85 (1975).

[8] A.M. Polyakov, Nucl. Phys. **B120**, 429 (1977).

[9] C.G. Callan, R.E. Dashen, and D.J. Gross, Phys. Rev. **D17**, 2717 (1978).

[10] A.G. Williams, Nucl. Phys. B Proc. Suppl. **109A**, 141 (2002).

[11] I.M. Singer, Phys. Scripta **24**, 817 (1981).

[12] I.M. Singer, Commun. Math. Phys. **64**, 7 (1978).

[13] K. Haller, in *Non-Perturbative QCD: Proceedings*, eds. H.M. Fried, Y. Gabellini, and B. Muller (World Scientific, Singapore, 2002).

[14] V.N. Gribov, Nucl. Phys. **B139**, 1 (1978); M.B. Halpern and J. Koplik, Nucl. Phys. **B132**, 239 (1978).

[15] G. Esposito, *Dirac Operators and Spectral Geometry* (hep-th/9704016; Cambridge University Press, Cambridge, 1998).

[16] P. van Baal and R.E. Cutkosky, in *Tianjin 1992, Proceedings, Differential Geometric Methods in Theoretical Physics*; P. van Baal, Nucl. Phys. **B369**, 259 (1992), hep-th/9511119, hep-th/9711070, hep-ph/0008206.

[17] D. Zwanziger, Nucl. Phys. **B209**, 336 (1982).

[18] G.F. Dell'Antonio and D. Zwanziger, Nucl. Phys. **B326**, 333 (1989).

[19] D. Zwanziger, Nucl. Phys. **B323**, 513 (1989).

[20] R. Ticciati, *Quantum Field Theory for Mathematicians* (Cambridge University Press, Cambridge, 1999).

[21] A. Smilga, *Lectures on Quantum Chromodynamics* (World Scientific, Singapore, 2001).

[22] P. Bergmann and A. Komar, Int. J. Theor. Phys. **5**, 15 (1972).

[23] L. Lusanna and M. Pauri, gr-qc/0301040.

[24] J. Milnor, in *Relativity, Groups and Topology II*, eds. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

[25] L. Lusanna, in *Florence 2001, A Relativistic Spacetime Odyssey*, eds. I. Ciufolini, D. Dominici, and L. Lusanna (World Scientific, Singapore, 2003).

[26] S. Shanmugadhasan, J. Math. Phys. **14**, 677 (1973).

[27] P.B. Bergmann, Phys. Rev. **75**, 680 (1949); P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University (Dover, New York, 2001); M.S. Plyushchay and A.V. Razumov, Int. J. Mod. Phys. **A11**, 1427 (1996).

[28] V. Rivasseau, *From Perturbative to Constructive Renormalization* (Princeton University Press,

Princeton, 1991).

[29] M. Daniel and C.M. Viallet, Phys. Lett. **B76**, 458 (1978).

[30] C. Fleischhack, Commun. Math. Phys. **234**, 423 (2003).

[31] A. Ashtekar, in *Proceedings of the Second International Conference of Gravitation and Cosmology*, ed. C. Vishveshwara (Cambridge University Press, Cambridge, 1993); A. Ashtekar and J. Lewandowski, J. Math. Phys. **36**, 2170 (1995).

[32] G.B. Tupper and F.G. Scholtz, hep-th/9501089.

[33] F. Loran, hep-th/0203148.

[34] D. Zwanziger, Nucl. Phys. **B345**, 461 (1990).

[35] S.V. Shabanov and J.R. Klauder, Phys. Lett. **B456**, 38 (1999).

[36] S.V. Shabanov, Phys. Lett. **B463**, 263 (1999).

[37] J. Manjavidze, A. Sissakian, and V. Voronyuk, hep-th/0402214.

[38] D. Zwanziger, Phys. Rev. **D69**, 016002 (2004).

[39] H. Neuberger, Phys. Lett. **B183**, 337 (1987).

[40] K.G. Wilson, Phys. Rev. **D10**, 2445 (1974).

[41] M.L. Paciello, C. Parrinello, S. Petrarca, B. Taglienti, and A. Vladikas, Phys. Lett. **B276**, 163 (1992).

[42] L. Giusti, M.L. Paciello, S. Petrarca, B. Taglienti, and C. Parrinello, Int. J. Mod. Phys. **A16**, 3487 (2001).

[43] A. Cucchieri, Nucl. Phys. **B508**, 353 (1997).

[44] T.P. Killingback, Phys. Lett. **B138**, 87 (1984).

[45] Y. Makeenko, *Methods of Contemporary Gauge Theory* (Cambridge University Press, Cambridge, 2002).

[46] T. Cornelissen, Master Thesis (Institute for Theoretical Physics, University of Amsterdam, 2002).

[47] E. Marinari, C. Parrinello, and R. Ricci, Nucl. Phys. **B362**, 487 (1991).

[48] C. Parrinello, S. Petrarca, and A. Vladikas, Phys. Lett. **B268**, 236 (1991).

[49] A. Cucchieri and T. Mendez, Nucl. Phys. B Proc. Suppl. **63**, 841 (1998).

[50] S. Hioki et al., Phys. Lett. **B271**, 201 (1991).

[51] G.S. Bali, V. Bornyakov, M. Müller–Preussker, and F. Pahl, Nucl. Phys. B Proc. Suppl. **42**, 852 (1995).

[52] G.S. Bali et al., Phys. Rev. **D54**, 2863 (1996).

[53] A. Nakamura and M. Plewnia, Phys. Lett. **B255**, 274 (1991).

[54] P. De Forcrand, J.E. Hetrick, A. Nakamura, and M. Plewnia, Nucl. Phys. B Proc. Suppl. **20**, 194 (1991).

[55] V.G. Bornyakov, V.K. Mitrjushkin, M. Müller–Preussker, and F. Pahl, Phys. Lett. **B317**, 596 (1993).

[56] P. de Forcrand and J.E. Hetrick, Nucl. Phys. B Proc. Suppl. **42**, 861 (1995).

[57] A. Nakamura and R. Sinclair, Phys. Lett. **B234**, 396 (1990).

[58] A. Nakamura, T. Saito, and S. Sakai, Phys. Rev. **D69**, 014506 (2004).

[59] M. Testa, Phys. Lett. **B429**, 349 (1998).